University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Water surface estimation

Daniel Wildt

SWARM Summer School
15 – 26 November 2021

16th November 2021

Outline I

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Ordinary differential equations

- Analytical solution
- Numerical solution

Water surface estimation

- Differential equation of the water surface
- Water surface at non-uniform flow
- Bifurcation

Retention basin

Ordinary differential equations

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

initial-value problem:

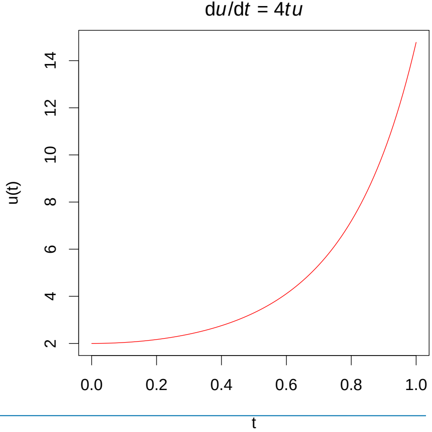
$$\frac{\mathrm{d}u}{\mathrm{d}t} = u' = f(t, u) \qquad (1)$$

initial condition:

$$u(0) = u_0 \qquad (2)$$

example:

$$u' = 4tu \qquad u(0) = 2 \qquad (3)$$



Ordinary differential equations: Analytical solution

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 4tu \Rightarrow \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}t} = 4t \Rightarrow \frac{1}{u}\mathrm{d}u = 4t\mathrm{d}t \Rightarrow \int \frac{1}{u}\mathrm{d}u = \int 4t\mathrm{d}t$$

$$\Rightarrow \ln u + C_1 = 2t^2 + C_2 \Rightarrow e^{\ln u} = e^{2t^2} \cdot \underbrace{e^{C_2 - C_1}}_{c} \tag{4}$$

$$\Rightarrow u = c \cdot e^{2t^2} \tag{5}$$

initial condition:

$$u(0) = c \cdot e^{2 \cdot 0^2} = 2 \Rightarrow c = 2$$

$$\Rightarrow u(t) = 2e^{2t^2} \tag{6}$$

Ordinary differential equations: Analytical solution

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

"test": Derivative of equation (6) and comparison to (3):

$$u' = 2 \cdot e^{2t^2} \cdot 4t = 4t \cdot \underbrace{2e^{2t^2}}_{=u(t)}$$

$$u' = 4t \cdot u$$

Ordinary differential equations: Numerical solution

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Explicit Euler method:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u' = 4tu \Rightarrow \frac{\Delta u}{\Delta t} = 4tu \Rightarrow \frac{u_{n+1} - u_n}{\underbrace{t_{n+1} - t_n}_{h}} = 4t \cdot u(t_n) \tag{7}$$

$$\Rightarrow u_{n+1} = u_n + 4 \cdot h \cdot t \cdot u_n \tag{8}$$

Ordinary differential equations: Numerical solution

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Predictor-Corrector method (Runge-Kutta method):

$$\frac{\mathrm{d}u}{\mathrm{d}t} = u' = 4tu \Rightarrow \frac{\Delta u}{\Delta t} = 4tu \Rightarrow \frac{u_{n+1} - u_n}{\underbrace{t_{n+1} - t_n}_{h}} = 4t_{n+\frac{1}{2}} \cdot u(t_{n+\frac{1}{2}}) \tag{9}$$

$$\Rightarrow u_{n+1} = u_n + 4 \cdot h \cdot t_{n+\frac{1}{2}} \cdot u_{n+\frac{1}{2}} \tag{10}$$

$$u_{n+\frac{1}{2}} = \frac{u_n + u^*_{n+1}}{2} \qquad t_{n+\frac{1}{2}} = \frac{t_n + t^*_{n+1}}{2} \tag{11}$$

u^*_{n+1} is estimatied using the Euler-method (8)

Ordinary differential equations: Numerical solution

University of Natural Resources
and Life Sciences, Vienna
rtment of Water, Atmosphere
Environment

	A	B
1	u0	2
2	h	0.1

(a) input

	A	B	C	D
1	n	t	u explicit Euler	u analytical
2	0	0	2.0000	2.0000
3	1	0.1	2.0800	2.0404
4	2	0.2	2.2464	2.1666
5	3	0.3	2.5160	2.3944
6	4	0.4	2.9185	2.7543
7	5	0.5	3.5022	3.2974
8	6	0.6	4.3428	4.1089
9	7	0.7	5.5587	5.3289
10	8	0.8	7.3375	7.1933
11	9	0.9	9.9790	10.1062
12	10	1	13.9707	14.7781

(b) solution using the Euler-method

Fig.: Solution of the ordinary differential equation (3) using the Euler method for discretization

Ordinary differential equations: Numerical solution

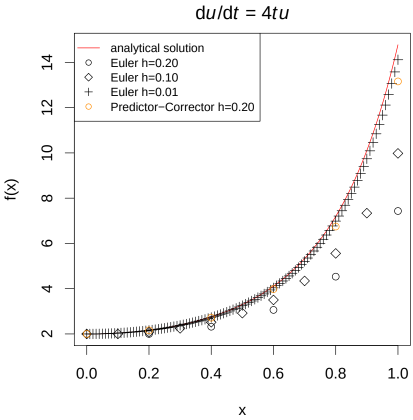
University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

	A	B	C	D	E	F	G
1	n	t	u_n Prädiktor-Ko	u_(n+1)*	u_(n+0.5)	t_(n+0.5)	u_(n+1)
2	0	0	2.0000	2.0000	2.0000	0.0500	2.0400
3	1	0.1	2.0400	2.1216	2.0808	0.1500	2.1648
4	2	0.2	2.1648	2.3380	2.2514	0.2500	2.3900
5	3	0.3	2.3900	2.6768	2.5334	0.3500	2.7447
6	4	0.4	2.7447	3.1838	2.9642	0.4500	3.2782
7	5	0.5	3.2782	3.9339	3.6061	0.5500	4.0716
8	6	0.6	4.0716	5.0487	4.5601	0.6500	5.2572
9	7	0.7	5.2572	6.7292	5.9932	0.7500	7.0552
10	8	0.8	7.0552	9.3128	8.1840	0.8500	9.8377

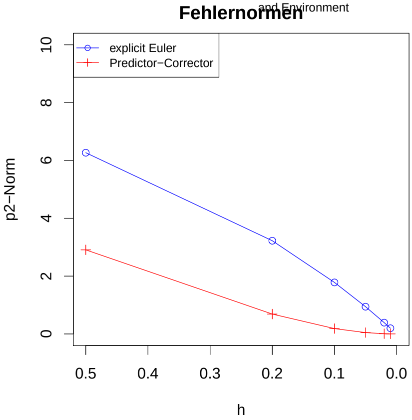
Fig.: Solution of the ordinary differential equation (3) using the Predictor-Corrector method for discretization

Ordinary differential equations: Numerical solution

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment



(a) solution



(b) $p2$-norms

Water surface estimation: Differential equation of the water surface

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{I_S - I_E}{1 + \frac{\mathrm{d}h_g}{\mathrm{d}h}} = \frac{I_S - I_E}{1 - \frac{Q^2 \cdot B}{g \cdot A^3}} = \frac{I_S - I_E}{1 - Fr^2} \tag{12}$$

see e. g. Maurer (2010) for the derivation

Analytical solution available only for a small number of cases, e. g.:

- method by Tolkmitt for parabolic cross sections
- method by Rühlmann for rectangular cross sections

Water surface estimation: Differential equation of the water surface

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Finite Differences Method:

$$\frac{h_{i+1} - h_i}{x_{i+1} - x_i} = \frac{I_S - I_E}{1 - \frac{Q^2 \cdot B}{g \cdot A^3}} \qquad \Rightarrow h_{i+1} = h_i + \Delta x \cdot \frac{I_S - I_E}{1 - \frac{Q^2 \cdot B}{g \cdot A^3}} \tag{13}$$

slope of the energy grade line I_E, e. g. from Manning-Strickler equation:

$$Q(h) = k_{\mathrm{St}} \cdot I_E^{\frac{1}{2}} \cdot R(h)^{\frac{2}{3}} \cdot A(h) \Rightarrow I_E = \frac{Q^2(h)}{k_{\mathrm{St}}^2 \cdot R(h)^{\frac{4}{3}} \cdot A^2(h)} \tag{14}$$

Euler scheme:

$$I_E = I_{Ei} = \frac{Q^2}{k_{\mathrm{St}}^2 \cdot R(h_i)^{\frac{4}{3}} \cdot A^2(h_i)} \tag{15}$$

Water surface estimation: Differential equation of the water surface

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Predictor-Corrector scheme:

$$I_E = I_{Ei+\frac{1}{2}} = \frac{Q^2}{k_{\mathrm{St}}^2 \cdot R(h_{i+\frac{1}{2}})^{\frac{4}{3}} \cdot A^2(h_{i+\frac{1}{2}})} \tag{16}$$

using:

$$A(h_{i+\frac{1}{2}}) = \frac{A(h_i) + A(h_{i+1}^*)}{2} \qquad R(h_{i+\frac{1}{2}}) = \frac{R(h_i) + R(h_{i+1}^*)}{2} \tag{17}$$

h_{n+1}^* estimatied using the Euler scheme (8)

Water surface estimation: Differential equation of the water surface

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

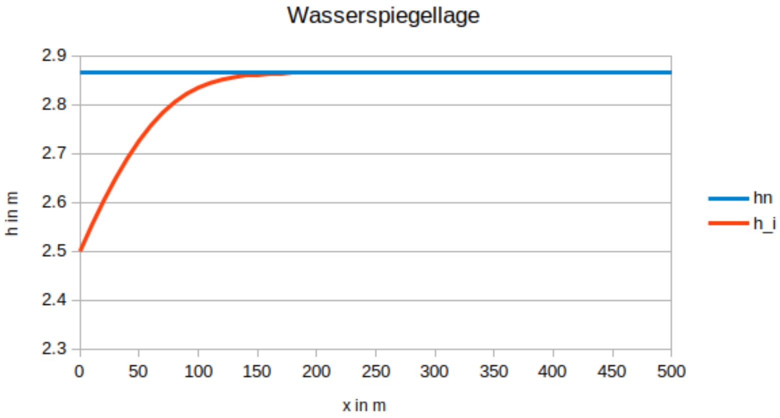
	A	B	C	D
4				
5	Erdbeschleunigung	g	m/s^2	9.81
6	Flussbreite	b	m	30
7	Wasserstand am Ausgangspunkt	h0	m	2.5
8	Abfluss	Q	m^3/s	525
9	Stricklerbeiwert	kST	m^(1/3)/s	34
10	Sohlgefälle	Is	1	0.01
11	Normalabflusstiefe	hn	m	2.86624
12				
13	Diskretisierungsschritt	Delta x	m	10

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	i	x	h_i	U_i	A_i	R_i	IE_i	h_(i+1)	h_(i+1)-h_i	h_(i+1)/h_i	h_(i+1)-hn	h_(i+1)/hn	hn
2		m	m	m	m^2	m	1	m	m	1	m	1	m
3	0	0.0	2.5000	35.0000	75.0000	2.1429	0.0153	2.55354	0.05354	1.0214	-0.3127	0.8909	2.86624
4	1	10.0	2.5535	35.1071	76.6062	2.1821	0.0144	2.60332	0.04978	1.0195	-0.2629	0.9083	2.86624
5	2	20.0	2.6033	35.2066	78.0996	2.2183	0.0135	2.64896	0.04564	1.0175	-0.2173	0.9242	2.86624
6	3	30.0	2.6490	35.2979	79.4687	2.2514	0.0128	2.69009	0.04113	1.0155	-0.1761	0.9385	2.86624
7	4	40.0	2.6901	35.3802	80.7027	2.2810	0.0122	2.72640	0.03632	1.0135	-0.1398	0.9512	2.86624
8	5	50.0	2.7264	35.4528	81.7921	2.3071	0.0117	2.75770	0.03129	1.0115	-0.1085	0.9621	2.86624
9	6	60.0	2.7577	35.5154	82.7309	2.3294	0.0113	2.78392	0.02622	1.0095	-0.0823	0.9713	2.86624

Fig.: Solution of the differential equation of the water surface using the Euler scheme

Water surface estimation: Differential equation of the water surface

University of Natural Resources and Life Sciences, Vienna



Water surface estimation: Differential equation of the water surface

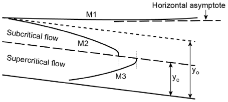
University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	i	x	h_i	U_i*	A_i*	R_i*	IE_i*	h_(i+1)*	h_(i+1/2)	U_i	A_i	R_i	IE_i	h_(i+1)	h_(i+1)-h_i	h_(i+1)/h_i	h_(i+1)/hn
2		m	m	m	m²	m	1	m	m	m	m²	m	1	m	m	1	1
3	0	0.0	2.5000	35.0000	75.0000	2.1429	0.0153	2.5535	2.5268	35.0535	75.8031	2.1625	0.0148	2.5517	0.0517	1.0207	0.8903
4	1	10.0	2.5517	35.1035	76.5521	2.1808	0.0144	2.6017	2.5767	35.1534	77.3009	2.1990	0.0140	2.5997	0.0479	1.0188	0.9070
5	2	20.0	2.5997	35.1994	77.9903	2.2157	0.0136	2.6456	2.6227	35.2453	78.6798	2.2323	0.0132	2.6435	0.0438	1.0169	0.9223
6	3	30.0	2.6435	35.2870	79.3050	2.2474	0.0129	2.6852	2.6644	35.3287	79.9307	2.2625	0.0126	2.6829	0.0394	1.0149	0.9360
7	4	40.0	2.6829	35.3658	80.4877	2.2759	0.0123	2.7201	2.7015	35.4031	81.0459	2.2892	0.0120	2.7177	0.0348	1.0130	0.9482
8	5	50.0	2.7177	35.4355	81.5323	2.3009	0.0118	2.7503	2.7340	35.4681	82.0208	2.3125	0.0116	2.7479	0.0301	1.0111	0.9587
9	6	60.0	2.7479	35.4957	82.4362	2.3224	0.0114	2.7758	2.7618	35.5236	82.8547	2.3324	0.0112	2.7734	0.0255	1.0093	0.9676
10	7	70.0	2.7734	35.5467	83.2010	2.3406	0.0111	2.7967	2.7851	35.5701	83.5516	2.3489	0.0109	2.7945	0.0211	1.0076	0.9750
11	8	80.0	2.7945	35.5889	83.8336	2.3556	0.0108	2.8136	2.8040	35.6080	84.1206	2.3624	0.0107	2.8115	0.0170	1.0061	0.9809
12	9	90.0	2.8115	35.6230	84.3449	2.3677	0.0106	2.8268	2.8192	35.6383	84.5748	2.3731	0.0105	2.8250	0.0135	1.0048	0.9856
13	10	100.0	2.8250	35.6500	84.7495	2.3773	0.0105	2.8370	2.8310	35.6620	84.9300	2.3815	0.0104	2.8355	0.0105	1.0037	0.9893
14	11	110.0	2.8355	35.6709	85.0636	2.3847	0.0103	2.8447	2.8401	35.6802	85.2027	2.3880	0.0103	2.8435	0.0080	1.0028	0.9920
15	12	120.0	2.8435	35.6869	85.3035	2.3903	0.0103	2.8505	2.8470	35.6939	85.4092	2.3928	0.0102	2.8495	0.0060	1.0021	0.9942
16	13	130.0	2.8495	35.6990	85.4843	2.3946	0.0102	2.8548	2.8521	35.7042	85.5636	2.3965	0.0102	2.8540	0.0045	1.0016	0.9957
17	14	140.0	2.8540	35.7079	85.6191	2.3978	0.0101	2.8579	2.8559	35.7119	85.6780	2.3991	0.0101	2.8573	0.0033	1.0012	0.9969

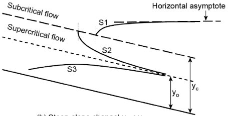
Fig.: Solution of the differential equation of the water surface using the Predictor-Corrector scheme

Water surface estimation: Water surface at non-uniform flow

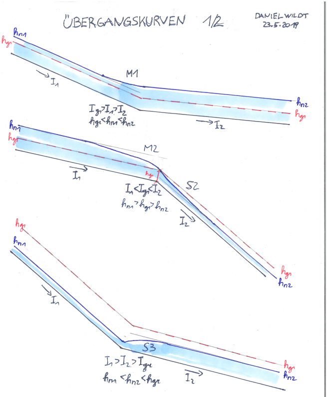
University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere

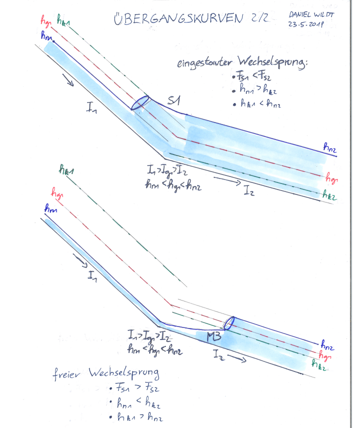


(a) Mild-slope channel $y_o > y_c$



(b) Steep-slope channel $y_o < y_c$





Water surface estimation: Water surface at non-uniform flow

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

subcritical uniform flow at:

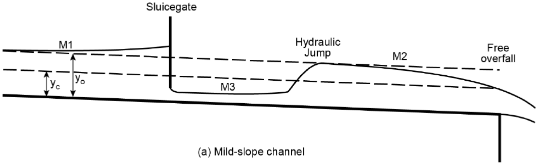
- M1 build-up at subcritical flow
- M2 acceleration of subcritical flow
- M3 build-up of supercritical flow at low bed slope upstream the hydraulic jump

supercritical uniform flow:

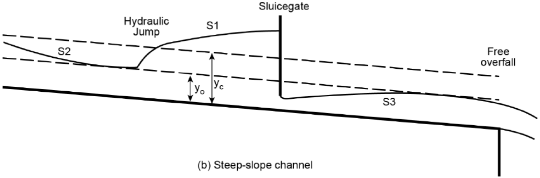
- S1 build-up of supercritical flow at low bed slope downstream the hydraulic jump
- S2 acceleration of supercritical flow
- S3 build-up at supercritical flow

Water surface estimation: Water surface at non-uniform flow

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment



(a) Mild-slope channel



(b) Steep-slope channel

Water surface estimation: Water surface at non-uniform flow

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Use the MS Excel workbook `Staukurve_v10.xls` to calculate the water surface at non-uniform flow conditions given the following values:

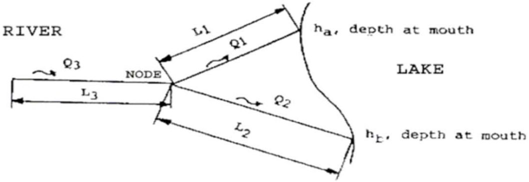
- river width $b = 30\,\mathrm{m}$
- water depth at the starting point $h_0 = 2{,}5\,\mathrm{m}$
- discharge $Q = 525\,\mathrm{m}^3\,\mathrm{s}^{-1}$
- Manning's coefficient $\frac{1}{n} = k_{\mathrm{St}} = 34\,\mathrm{m}^{1/3}\,\mathrm{s}^{-1}$
- bed slope $I = 0{,}01$

Tasks `Staukurve_v10.xls`

- Which type of curve will be found under the given conditions?
- Change the input values for other possible curve types!

Water surface estimation: Bifurcation

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment



Dx =	1000 m
C =	50 $m^{1/2}/s$
α=	150 m^2/s

L_1 =	15 km
L_2 =	25 km
L_3 =	10 km
b_1 =	60 m
b_2 =	72 m
b_3 =	120 m
i_1 =	5E-04
i_2 =	3E-04
i_3 =	5E-04
$h_A = h_B$ =	5 m
Q =	750 m^3/s

Fig.: Backwater curve and bifurcation, solve using `Backwater_cure_v11.xls`.
solution: a) $Q_1 = 388{,}7\,\mathrm{m}^3\,\mathrm{s}^{-1}$, $Q_2 = 361{,}3\,\mathrm{m}^3\,\mathrm{s}^{-1}$; b) 3,15 m and 3,23 m

Retention basin

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

$$dV = (Q_{in} - Q_{aus})\ dt \quad (18)$$

$$V = A \cdot h \quad (19)$$

$$dV = \frac{A\ dh}{dt} = (Q_{in} - Q_{out})\ dt \quad (20)$$

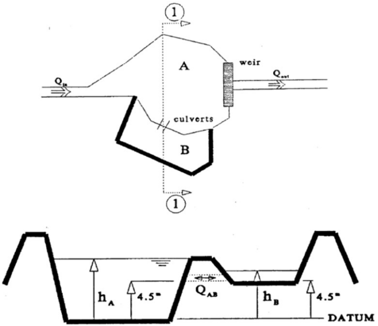


Fig.: Scheme of the retention basin

Retention basin: Discretization basin A

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Euler method:

$$\Delta V_{\mathrm{A}} = A_{\mathrm{A}} \cdot \Delta h_{\mathrm{A}} = (Q_{\mathrm{in}} - Q_{\mathrm{out}} - Q_{\mathrm{AB}}) \cdot \Delta t \qquad (21)$$

$$\left(h_{\mathrm{A}}^{n+1} - h_{\mathrm{A}}^{n}\right) \cdot A_{\mathrm{A}} = (Q_{\mathrm{in}} - Q_{\mathrm{out}} - Q_{\mathrm{AB}})^{n} \cdot \Delta t \qquad (22)$$

$$\Rightarrow h_{\mathrm{A}}^{n+1} = \frac{\Delta t}{A_{\mathrm{A}}} \cdot (Q_{\mathrm{in}} - Q_{\mathrm{out}} - Q_{\mathrm{AB}})^{n} + h_{\mathrm{A}}^{n} \qquad (23)$$

Predictor-Corrector method:

$$\Rightarrow h_{\mathrm{A}}^{n+1} = \frac{\Delta t}{A_{\mathrm{A}}} \cdot (Q_{\mathrm{zu}} - Q_{\mathrm{out}} - Q_{\mathrm{AB}})^{n+\frac{1}{2}} + h_{\mathrm{A}}^{n+\frac{1}{2}} \qquad (24)$$

Retention basin: Discretization basin B

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Euler method:

$$\Delta V_{\mathrm{B}} = A_{\mathrm{B}} \cdot \Delta h_{\mathrm{B}} = Q_{\mathrm{AB}} \cdot \Delta t \tag{25}$$

$$\left(h_{\mathrm{B}}^{n+1} - h_{\mathrm{B}}^{n}\right) \cdot A_{\mathrm{B}} = Q_{\mathrm{AB}}^{n} \cdot \Delta t \tag{26}$$

$$\Rightarrow h_{\mathrm{B}}^{n+1} = \frac{\Delta t}{A_{\mathrm{B}}} \cdot Q_{\mathrm{AB}}^{n} + h_{\mathrm{B}}^{n} \tag{27}$$

Predictor-Corrector method:

$$\Rightarrow h_{\mathrm{B}}^{n+1} = \frac{\Delta t}{A_{\mathrm{B}}} \cdot Q_{\mathrm{AB}}^{n+\frac{1}{2}} + h_{\mathrm{B}}^{n+\frac{1}{2}} \tag{28}$$

Retention basin: Outlet flow rate

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Flow rate Q_{out} through outlet basin A:

$$Q_{\mathrm{out}} = \min\left(12{,}5 \cdot h_{\mathrm{A}}^{\frac{3}{2}}, 100\,\mathrm{m}^3\,\mathrm{s}^{-1}\right) \tag{29}$$

$$Q^0_{\mathrm{out}} = 35\,\mathrm{m}^3\,\mathrm{s}^{-1} \qquad \Rightarrow h^0_{\mathrm{A}} = \left(\frac{Q^0_{\mathrm{out}}}{12{,}5}\right)^{\frac{2}{3}} = 1{,}99\,\mathrm{m} \tag{30}$$

Flow rate between basin A and B:

$$A_{\mathrm{AB}} = \begin{cases} 36{,}5 \cdot \sqrt{h_{\mathrm{A}} - h_{\mathrm{B}}} & \text{für} \quad h_{\mathrm{A}} > h_{\mathrm{B}} \geq 4{,}5\,\mathrm{m} \\ -36{,}5 \cdot \sqrt{h_{\mathrm{B}} - h_{\mathrm{A}}} & \text{für} \quad h_{\mathrm{B}} > h_{\mathrm{A}} \text{ und } h_{\mathrm{B}} > 4{,}5\,\mathrm{m} \\ 0\,\mathrm{m}^3\,\mathrm{s}^{-1} & \text{für} \quad 4{,}5\,\mathrm{m} > h_{\mathrm{B}} \geq h_{\mathrm{A}} \end{cases} \tag{31}$$

Retention basin

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

A system of two retention basins A and B is installed for flood protection (see figure).

Tasks `Rueckhaltebecken_v11.xls`

1. What time does it take until steady flow conditions are reached after the beginning of the given event?
   - 1.a using the default basin sizes ($A_\mathrm{A} = 10^6\,\mathrm{m}^2$, $A_\mathrm{B} = 500\,000\,\mathrm{m}^2$)
   - 1.b basin A is 20 % bigger
   - 1.c basin B is 50 % smaller (basin A back to default size)
2. How can the differences be explained?

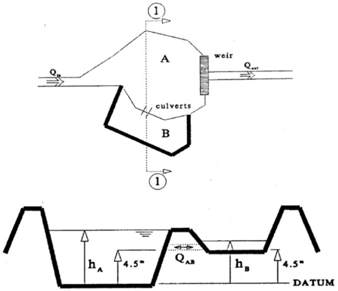


Fig.: Scheme of the retention basin

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

University of Natural Resources and Life Science, Vienna

Department of Water, Atmosphere and Environment
Institut of Hydraulic Engineering and River Research

Daniel Wildt, MSc

Muthgasse 107, A - 1190 Wien
Tel.: 01-47654-81935
daniel.wildt@boku.ac.at
http://www.wau.boku.ac.at/iwa/

Literatur I

University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Maurer, D. (2010). 'Arbeitsblattbasierende Programme zur eindimensionalen Berechnung von Staukurven, Schadstoffeinträgen und Hochwasserwellen in offenen Gerinnen'. Masterarbeit. Universität für Bodenkultur Wien.