University of Natural Resources and Life Sciences, Vienna
Department of Water, Atmosphere and Environment

Error estimation

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Outline I

University of Natural Resources
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Department of Water, Atmosphere
and Environment

Explicit scheme wrap-up

UNDA

Gaussian error estimation

- Volume measurement
- Flow measurement
- Thompson weir

Explicit scheme wrap-up

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Stencil of an explicit and an implicit scheme

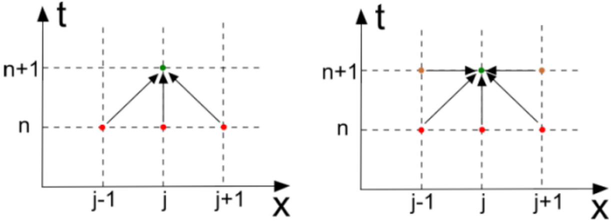

Fig.: Comparison of an explicit (left) and and implicit (right) scheme

Explicit scheme wrap-up

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Example: Advection equation

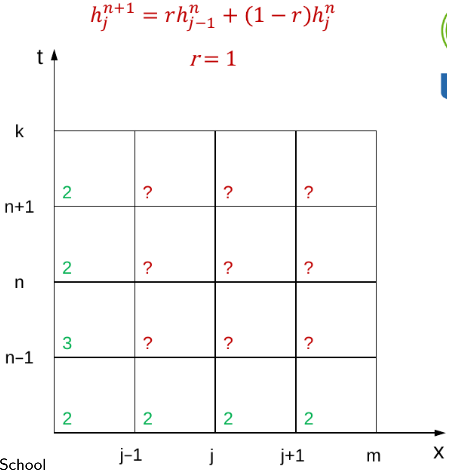
$$\frac{\partial h}{\partial t} + u \cdot \frac{\partial h}{\partial x} = 0 \qquad (1)$$

Discretization (upwind):

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} + u \cdot \frac{h_i^n - h_{i-1}^n}{\Delta x} = 0$$

$$\Rightarrow h_i^{n+1} = h_i^n - \underbrace{\frac{\Delta t \cdot u}{\Delta x}}_{r} \cdot (h_i^n - h_{h-1}^n)$$

$$\Rightarrow h_i^{n+1} = r \cdot h_{i-1}^n + (1 - r) \cdot h_i^n$$



Explicit scheme wrap-up

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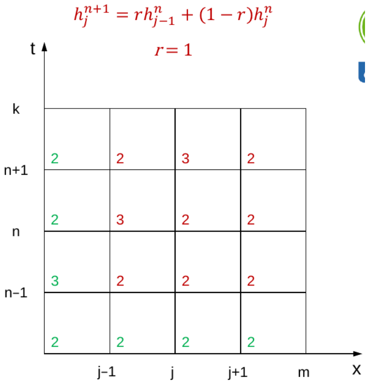
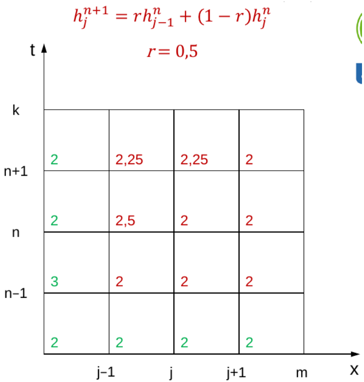


(a) $r = 1{,}0$



(b) $r = 0{,}5$

UNDA

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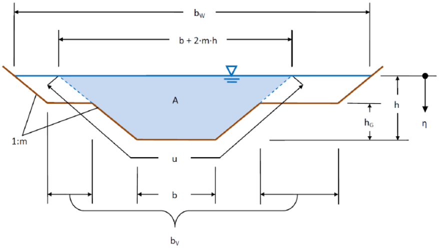
- UNDA (lat. the wave) is a MS Excel worksheet based programm for 1D flow calculation in open channels;
- developed by Aubrunner (2009) in the course of his Diploma-thesis at BOKU
- numerical solution of the Saint-Venant equations using the Preissmann-Scheme



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- river networks including retention basins, lakes, weirs and connections
- generation of inlet boundary condition a. o. form a unit-hydrograph

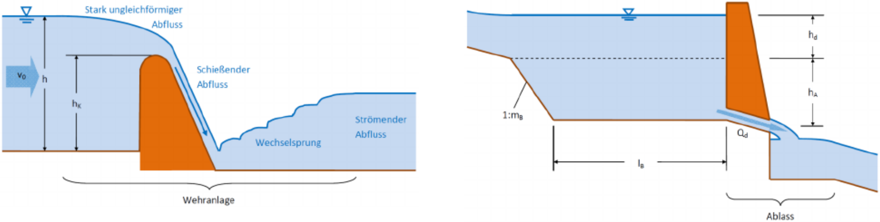

Fig.: boundary conditions for weirs (Aubrunner, 2009)

UNDA: program start

Unda, die Welle

Um dieses Programm nützen zu können, müssen Makros aktiviert sein!

Neue Mappe	Mappe öffnen

Berechnungen können nur durchgeführt werden, wenn im Vordergrund eine Mappe mit Daten und im Hintergrund diese Mappe geöffnet ist!

Menü öffnen: Strg + Umsch + U

Shortcuts

Strg + Umsch +	Aktion
B	*Welle Berechnen*
R	*Gerinne-Rechner*
Z	*Welle Zeichnen*
A	*Welle Animieren*
G	*Ganglinie/Pegelschlüssel Zeichnen*
V	*Vorschau*
X	*Grafik Schließen*
N	*Neue Mappe*
O	*Mappe Öffnen*
S	*Mappe Speichern*
K	*Mappe Kopieren*
D	*Dokumentation*
I	*Info*

Fig.: `Unda2003_v4.xls`

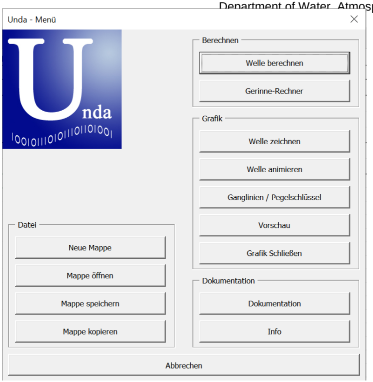

Fig.: start view UNDA (Aubrunner, 2009)

UNDA: Example

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Simulation time and calculation:

- simulation time: 1 800 s
- 900 time steps
- max. 10 iterations for each time step
- 6 significant digits

output:

- 10 time steps
- 100 m cross section
- 3 digits

river:

- length: 3 km
- 1000 intervals

cross section geometry:

- width: 3 m
- bankslope: 2
- roughness: $k_{\mathrm{St}} = 50\,\mathrm{m}^{1/3}\,\mathrm{s}^{-1}$

UNDA: Example

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Unit hydrograph for upper boundary:

- baseflow: $Q_B = 10\,\mathrm{m^3\,s^{-1}}$
- start of precipitation at minute 0
- precipitation duration: 10 min
- storage coefficient: 100 s
- number of storages: 3

outlet into the sea:

- water surface above river bed: 1,2 m

possible extension:

- implementation of a retention basin
- connection of an additional river and superposition of flows
- ...

Gaussian error estimation

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Average error of a variable F based on independent variables x_i:

$$\bar{\Delta F} = \sqrt{\sum_i \left(\frac{\partial F}{\partial x_i} \cdot \sigma_{x_i} \right)^2} \tag{2}$$

Maximum error:

$$\Delta F = \sum_i \left| \frac{\partial F}{\partial x_i} \right| \cdot \sigma_{x_i} \tag{3}$$

pre-assumptions:

- normal distributed errors
- small standard deviations $\sigma_{x_i} \ll x_i$

Gaussian error estimation: Volume measurement

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Example: Calculation of the volume of a hexahedra from the length of its three side; expected error from length measurements 1 ‰ , but 2 mm minimum

- $l = 6\,\text{m} \rightarrow \Delta l = \pm 6\,\text{mm}$
- $b = 1\,\text{m} \rightarrow \Delta b = \pm 2\,\text{mm}$
- $h = 2\,\text{m} \rightarrow \Delta h = \pm 2\,\text{mm}$

$$V = l \cdot b \cdot h \qquad \rightarrow \frac{\partial V}{\partial l} = b \cdot h \tag{4}$$

$$\rightarrow \frac{\partial V}{\partial b} = l \cdot h \tag{5}$$

$$\rightarrow \frac{\partial V}{\partial h} = l \cdot b \tag{6}$$

Gaussian error estimation: Volume measurement

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Average relative error (2)

$$\frac{\Delta \bar{V}}{V} = \frac{1}{V} \cdot \sqrt{\left(\frac{\partial V}{\partial l}\Delta l\right)^2 + \left(\frac{\partial V}{\partial b}\Delta b\right)^2 + \left(\frac{\partial V}{\partial h}\Delta h\right)^2} \tag{7}$$

$$= \sqrt{b^2 h^2 \cdot \Delta l^2 \cdot \frac{1}{l^2 b^2 h^2} + l^2 h^2 \cdot \Delta b^2 \cdot \frac{1}{l^2 b^2 h^2} + l^2 b^2 \cdot \Delta h^2 \cdot \frac{1}{l^2 b^2 h^2}} \tag{8}$$

$$= \sqrt{\frac{\Delta l^2}{l^2} + \frac{\Delta b^2}{b^2} + \frac{\Delta h^2}{h^2}} \tag{9}$$

$$= \sqrt{0{,}001^2 + 0{,}002^2 + 0{,}001^2} = \underline{\underline{0{,}00245}}$$

Gaussian error estimation: Volume measurement

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Maximum error (3)

$$\Delta V = \left|\frac{\partial V}{\partial l}\right| \Delta l + \left|\frac{\partial V}{\partial b}\right| \Delta b + \left|\frac{\partial V}{\partial h}\right| \Delta h \tag{10}$$

$$= bh \cdot \Delta l + lh \cdot \Delta b + lb \cdot \Delta h \tag{11}$$

$$= 2 \cdot 0{,}006 + 12 \cdot 0{,}002 + 6 \cdot 0,002 = \underline{\underline{0{,}048\,\mathrm{m}^3}} \rightarrow \frac{\Delta V}{V} = 0{,}004$$

Gaussian error estimation: Flow measurement

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Example: Use of the hexahedral container from the example above for flow measurement $Q = \frac{V}{t}$:

- Accuracy of the volume as above
- time measurement
  a. $t = 22{,}4\,\mathrm{s}$ with an estimated error of $0{,}5\,\mathrm{s}$
  b. six repetitions of the time measurement: $22{,}4\,\mathrm{s}$, $22{,}1\,\mathrm{s}$, $22{,}2\,\mathrm{s}$, $22{,}8\,\mathrm{s}$, $22{,}0\,\mathrm{s}$, $22{,}5\,\mathrm{s} \rightarrow \bar{t} = 22{,}333\,\mathrm{s}$

estimate the average and maximum relative error of the flow measurement

Gaussian error estimation: Flow measurement

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Use of a hexahedral container for flow measurement:

$$Q = \frac{V}{t} \qquad \rightarrow \frac{\partial Q}{\partial V} = \frac{1}{t} \tag{12}$$

$$\rightarrow \frac{\partial Q}{\partial t} = -\frac{V}{t^2} \tag{13}$$

$$\Delta Q = \sqrt{\left(\frac{\partial Q}{\partial V}\Delta V\right)^2 + \left(\frac{\partial Q}{\partial t}\Delta t\right)^2} = \sqrt{\frac{1}{t^2}\Delta V^2 + \frac{V^2}{t^4}\Delta t^2} \tag{14}$$

Gaussian error estimation: Flow measurement

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Average relative error:

$$\frac{\Delta \bar{Q}}{Q} = \frac{1}{Q} \cdot \sqrt{\frac{1}{t^2}\Delta V^2 + \frac{V^2}{t^4}\Delta t^2} \tag{15}$$

$$= \sqrt{\frac{1}{\cancel{t^2}}\Delta V^2 \frac{\cancel{t^2}}{V^2} + \frac{\cancel{V^2}}{t^4}\Delta t^2 \frac{t^2}{\cancel{V^2}}} = \sqrt{\frac{\Delta V^2}{V^2} + \frac{\Delta t^2}{t^2}} \tag{16}$$

Maximum error:

$$\Delta Q = \left|\frac{\partial Q}{\partial V}\right| \Delta V + \left|\frac{\partial Q}{\partial t}\right| \Delta t = \left|\frac{1}{t}\right| \Delta V + \left|\frac{V}{t^2}\right| \Delta t \tag{17}$$

Gaussian error estimation: Flow measurement

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ad a.:

average relative error from equation (16)

$$\frac{\bar{\Delta Q}}{Q} = \sqrt{\frac{\Delta V^2}{V^2} + \frac{\Delta t^2}{t^2}} = \sqrt{0{,}00245^2 + \left(\frac{0{,}5}{22{,}4}\right)^2} = \underline{\underline{0{,}0225}}$$

maximum relative error from equation (17):

$$\Delta Q = \frac{1}{t}\Delta V + \frac{V}{t^2}\Delta t = \frac{1}{22{,}4} \cdot 0{,}048 + \frac{12}{22{,}4^2} \cdot 0{,}5 = 0{,}0141\,\mathrm{m^3\,s^{-1}}$$

$$\rightarrow \frac{\Delta Q}{Q} = \underline{\underline{0{,}026}}$$

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ad b.:

Standard deviation of time measurements:

$$\sigma_t = \Delta t = \sqrt{\frac{\sum_i (t_i - \overline{t})}{n-1}} = \sqrt{\frac{0{,}433}{5}} = 0{,}2944\,\mathrm{s} \tag{18}$$

relative error of the time measurement:

$$\frac{\Delta t}{\overline{t}} = \frac{0{,}2944}{22{,}333} = 0{,}01318 \tag{19}$$

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average relative error from equation (16)

$$\frac{\bar{\Delta Q}}{Q} = \sqrt{\frac{\Delta V^2}{V^2} + \frac{\Delta t^2}{t^2}} = \sqrt{0{,}00245^2 + 0{,}01318^2} = \underline{\underline{0{,}01341}}$$

maximum relative error from equation (17):

$$\Delta Q = \frac{1}{t}\Delta V + \frac{V}{t^2}\Delta t = \frac{1}{22{,}3} \cdot 0{,}048 + \frac{12}{22{,}3^2} \cdot 0{,}29 = 0{,}009\,\mathrm{m}^3\,\mathrm{s}^{-1}$$

$$\rightarrow \frac{\Delta Q}{Q} = \frac{0{,}009}{0{,}537} = \underline{\underline{0{,}017}}$$

Gaussian error estimation: Thompson weir

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Calibration of a Thompson weir:

$$Q = C_v \cdot \tan\alpha \cdot h^{\frac{5}{2}} \Rightarrow C_v = \frac{Q}{h^{\frac{5}{2}} \cdot \tan\alpha} \quad \rightarrow \frac{\partial C_v}{\partial Q} = \frac{1}{h^{\frac{5}{2}} \cdot \tan\alpha} \tag{20}$$

$$\rightarrow \frac{\partial C_v}{\partial h} = \frac{Q}{\tan\alpha} \cdot -\frac{5}{2} \cdot h^{-\frac{7}{2}} \tag{21}$$

Gaussian error estimation: Thompson weir

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average error of the coefficient:

$$\Delta\bar{C}_v = \sqrt{\left(\frac{1}{h^{\frac{5}{2}}\cdot\tan\alpha}\Delta Q\right)^2 + \left(-\frac{5}{2}\cdot\frac{Q}{h^{\frac{7}{2}}\cdot\tan\alpha}\Delta h\right)^2} \tag{22}$$

$$\frac{\Delta\bar{C}_v}{C_v} = \sqrt{\frac{\Delta Q^2}{\cancel{h^5}\cdot\cancel{\tan}^2\alpha}\frac{\cancel{h^5}\cdot\cancel{\tan}^2\alpha}{Q^2} + \frac{25}{4}\cdot\frac{\cancel{Q}^2}{h^7\cdot\cancel{\tan}^2\alpha}\Delta h^2\cdot\frac{h^5\cdot\cancel{\tan}^2\alpha}{\cancel{Q}^2}} \tag{23}$$

$$= \sqrt{\frac{\Delta Q^2}{Q^2} + \frac{25}{4}\frac{\Delta h^2}{h^2}} \tag{24}$$

Gaussian error estimation: Thompson weir

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maximum error of the coefficient:

$$\Delta C_v = \left| \frac{1}{h^{\frac{5}{2}} \cdot \tan\alpha} \right| \Delta Q + \left| -\frac{5}{2} \cdot \frac{Q}{h^{\frac{7}{2}} \cdot \tan\alpha} \right| \Delta h \tag{25}$$

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Literatur I

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Aubrunner, B. (2009). 'Ein arbeitsblattbasiertes Programm zur eindimensionalen Berechnung des instationären Abflusses in offenen Gerinnen'. Masterarbeit. Universität für Bodenkultur Wien.